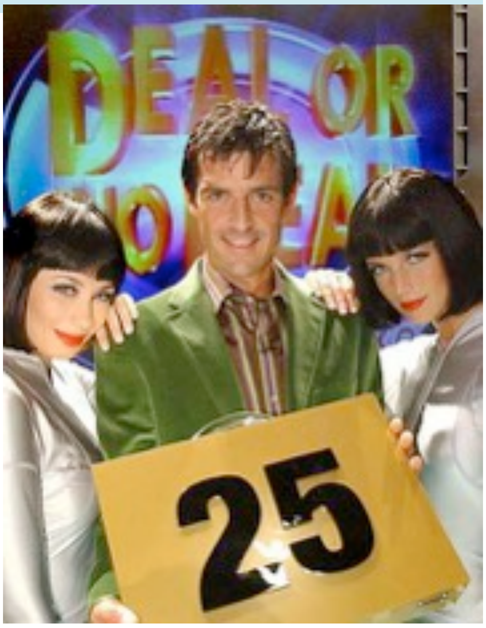
THE EXPECTED VALUE



Two famous examples to start with:

1. DEAL OR NO DEAL

In 2004 Swiss Television imported an entertaining programme from the Netherlands. There are 26 numbered suitcases with different but well known sums of money between CHF –.05 and CHF 250'000.–. The game starts with the guest picking one suitcase. Throughout the rest of the game the guest chooses one suitcase after the other and opens it. To produce some excitement from time to time a fictive bank in the background offers a certain amount of money to the guest in exchange of the first picked suitcase. The guest can accept the deal and walk out with the money offered - or he can go on playing confident that his suitcase contains more money. If the guest never accepts a deal and sticks to his suitcase he can take home the money that is in there.

Assuming that to the end of the game there are only four suitcases left: the one that had been picked first and three more with CHF 1.–, 5.–, 1,000.– and 100,000.–.
What would be a fair offer made by the bank?

From the bank's point of view there is a Laplace-experiment with the four possible outcomes 1, 5, 1,000 and 100,000 Swiss francs: $\Omega = \{1, 5, 1{,}000, 100{,}000\}$ (in francs).

Each outcome is equally likely: $p = \frac{1}{4}$

If the game came to this point 1,000 times it could be expected that each outcome would occur about 250 times. In 1,000 games Swiss Television would have to pay around (always in francs)

$$250 \cdot 1 + 250 \cdot 5 + 250 \cdot 1{,}000 + 250 \cdot 100{,}000$$

So the average win per game

$$= \frac{250 \cdot 1 + 250 \cdot 5 + 250 \cdot 1{,}000 + 250 \cdot 100{,}000}{1{,}000} = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 5 + \frac{1}{4} \cdot 1{,}000 + \frac{1}{4} \cdot 100{,}000 = 25{,}251.50$$

A little more general:
If the game came to this point **n** times it could be expected that each outcome would occur about $\frac{n}{4}$ times. In n games Swiss Television would have to pay around (always in francs)

$$\frac{n}{4}\cdot 1+\frac{n}{4}\cdot 5+\frac{n}{4}\cdot 1{,}000+\frac{n}{4}\cdot 100{,}000$$

So the average win per game

$$=\frac{\frac{n}{4}\cdot 1+\frac{n}{4}\cdot 5+\frac{n}{4}\cdot 1{,}000+\frac{n}{4}\cdot 100{,}000}{n}=\frac{1}{4}\cdot 1+\frac{1}{4}\cdot 5+\frac{1}{4}\cdot 1{,}000+\frac{1}{4}\cdot 100{,}000=25{,}251.50$$

Obviously the expected win does not depend on the number of games played.
In this case a fair offer by the bank would be **CHF 25,251.50**.

NB
The expected win only depends on the outcomes and on their probabilities:

$$\text{expected win}=\boxed{\frac{1}{4}}\cdot\boxed{1}+\boxed{\frac{1}{4}}\cdot\boxed{5}+\boxed{\frac{1}{4}}\cdot\boxed{1{,}000}+\boxed{\frac{1}{4}}\cdot\boxed{100{,}000}$$

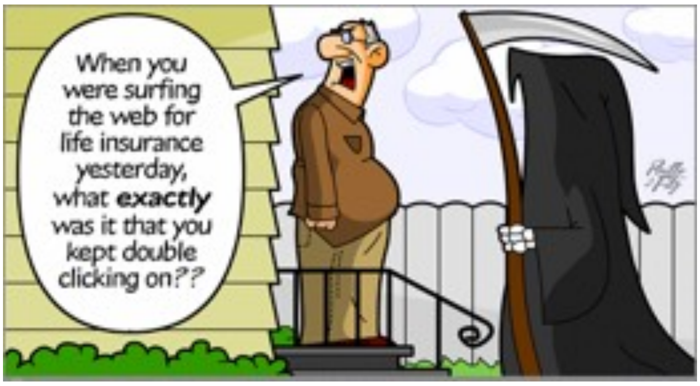
probabilities outcomes

A little bit of history:
DEAL OR NO DEAL: THE FIRST 20 WINS

programme	win	deal	suitcase	programme	win	deal	suitcase
1	**49,900**	X		**11**	**8,600**	X	
2	**46,500**	X		**12**	**2,500**	X	
3	**9,000**	X		**13**	**100,000**		X
4	**5,000**		X	**14**	**30,000**		X
5	**3,250**	X		**15**	**36,200**	X	
6	**15,000**		X	**16**	**24,000**	X	
7	**50,000**		X	**17**	**1,500**		X
8	**50**		X	**18**	**1,500**		X
9	**56,300**	X		**19**	**30,000**		X
10	**5,000**		X	**20**	**0.05**		X

2. INSURANCE PREMIUMS

A 64 year old man thinks about taking out a life insurance for the time until his retirement, i.e.for 1 year. If he should die within this year he wants his family to receive CHF 500,000.–, if he should survive they would get nothing. To keep things simple it is assumed that he pays the premium at the moment he takes out the insurance and that neither any interest nor any administration costs are taken into account.



Under these circumstances what should a fair premium be?

From the point of view of the insurance company it is a probability experiment with two possible outcomes:

1) The man dies, the insurance company gets the premium and has to pay CHF 500,000.–.
2) The man survives the year, the insurance company gets the premium and does not have to pay anything.

Let the premium be P. Therefore $\Omega = \{P, P - 500{,}000\}$.

What are now the probabilities of the two outcomes?

A mortality table is a table which shows, for each age, what the probability is that a person of that age will die before their next birthday. It is based on the data of a whole country over many years.
For a 64 year old man, for example, the mortality table tells a probability of 0.013262 to die before his 65th birthday.

$$p(P - 500{,}000) = 0.013262 \quad \text{and} \quad p(P) = 1 - 0.013262 = 0.986738$$

If the insurance company signs 1,000,000 contracts like this one, it must be expected that in 13262 cases the policy holder dies and the case P-500,000 occurs.
In all the other 986738 cases the policy holder survives and the case P occurs.

Average win per 1,000,000 contracts

$$= 13262 \cdot (P - 500{,}000) + 986738 \cdot P$$

Average win per contract

$$= \frac{13262 \cdot (P - 500{,}000) + 986738 \cdot P}{1{,}000{,}000} = 0.013262 \cdot (P - 500{,}000) + 0.986738 \cdot P$$

A little more general:
If the insurance company signs **n** contracts like this one, it must be expected that

Average win per n contracts

$$= 0.013262 \cdot n \cdot (P - 500{,}000) + 0.986738 \cdot n \cdot P$$

Average win per contract

$$= \frac{0.013262 \cdot n \cdot (P - 500{,}000) + 0.986738 \cdot n \cdot P}{n} = 0.013262 \cdot (P - 500{,}000) + 0.986738 \cdot P$$

Obviously the expected win does not depend on the number of contracts signed.

The deal is fair, if the expected win = 0.

$$\Leftrightarrow \quad 0.013262 \cdot (P - 500{,}000) + 0.986738 \cdot P = 0$$

$$\Leftrightarrow \quad P - 0.013262 \cdot 500{,}000 = 0$$

$$\Leftrightarrow \quad P = \mathbf{6{,}631}$$

So the insurance company needs for this insurance a premium of **CHF 6,631.–**.

NB
The expected win only depends on the outcomes and on their probabilities:

$$\text{expected win} = 0.013262 \cdot (P - 500{,}000) + 0.986738 \cdot P$$

probabilities outcomes

It is time to look at things in general.

DEFINITION

If a probability experiment has only got numerical outcomes, it is a **random variable**.

All random variables have in common that $\Omega = \{x_1, x_2, \dots, x_n\} \subset \mathbf{R}$.

With random variables some characteristic figures can be calculated.

DEFINITION

Let X be a random variable with $\Omega = \{x_1, x_2, \dots, x_n\}$.

Then the **expected value** $\mu = \sum_{i=1}^{n} p(x_i) \cdot x_i$.

The expected value is the average of all outcomes of the random variable weighted with their respective probability.

DEFINITION

Let X be a random variable with $\Omega = \{x_1, x_2, \dots, x_n\}$.

Then the **standard deviation** $\sigma = \sqrt{\sum_{i=1}^{n} p(x_i) \cdot (x_i - \mu)^2}$.

The standard deviation expresses how the values of the random variable X are spread around the expected value μ .

$$\sqrt{\sum_{i=1}^{n} p(x_i) \cdot (x_i - \mu)^2}$$

measures the distance between the values of X and μ

neglects the sign and gives values far away of μ more influence

adds the weight of the probability

$\sum$ adds up for all outcomes of X

$\sqrt{}$ gives σ the same unit as the outcomes have

σ is big	$\Rightarrow$	spread is big	$\Rightarrow$	risk is big
σ is small	$\Rightarrow$	spread is small	$\Rightarrow$	risk is small

1 Calculate the expected value and the standard deviation when rolling a die. Plot the probability distribution d.

2 Calculate the expected value and the standard deviation at roulette when

a) putting on colour.
b) putting on carré (four numbers arranged in a square on the tableau).
c) putting on a single number.

Plot the probability distribution each time.
Which strategy holds the biggest risk?

3 Calculate the premium for a 63 year old woman who wants to take out a life insurance for two years under the same conditions as in the 2nd example.

4 Calculate the bank's fair offer in the game show DEAL OR NO DEAL if there are the amounts of 250,000 and 150,000 and 30,000 and 10,000 and 5 francs still in the game.

5 A pair of dice are rolled. If this was done many times what would the average sum of pips be?

6 A friend offers a game: he tosses two coins of 1 franc, you toss one coin of 2 francs. The one with more "heads" wins all the coins.

a) Is the game fair?
b) Who takes the bigger risk?
c) Plot the corresponding probability distributions.